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# The structure of time-dependent correlation functions for classical liquids

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**Abstract.** A kinetic equation formulation is used to analyse the density and current correlation functions for a classical liquid. The matrix elements of the collision integral which contain information about the behaviour of the system in the short-time and hydrodynamic limits are eliminated formally from the kinetic equation and the correlation functions derived in terms of both these matrix elements and reduced correlation functions which are solutions of a kinetic equation with the remainder of the collision integral. This procedure facilitates the comparison of most of the existing theories and leads to a flexible scheme for developing interpolation formulae for the correlation functions which are valid in both the short-time and hydrodynamic limits.

## 1. Introduction

The time-dependent correlation functions, such as the coherent and incoherent scattering functions, give a complete description of the physical properties of classical liquids. As these functions can be measured experimentally by neutron scattering and computed directly by molecular dynamics, it is not surprising that they have received much attention from theorists during the last ten years (Schofield 1975). So far, a complete, unified theory for both the equilibrium and non-equilibrium properties of liquids is lacking and the equilibrium properties are usually assumed known and used in the calculation of the non-equilibrium functions. In spite of the diversity of theoretical methods (Schofield 1975), including linear response, mean-field and kinetic equation theories, the resulting correlation functions are often similar in structure although different in detail. Several theories attempt generalizations from a particular limit, the extreme limits being the hydrodynamic (Zwanzig and Bixon 1970) and short-time regions (Lebowitz *et al* 1969, Sykes 1973), and lead quite naturally to correlation functions which are valid in the limit but break down elsewhere. In this paper we examine the general structure of the time-dependent correlation functions and show what properties a theory must satisfy for the correlation functions to be correct in both the hydrodynamic and short-time limits.

We will confine our attention to the coherent or total functions and use a kinetic equation description with a collision integral  $\Sigma$ . The conditions which  $\Sigma$  must satisfy so that the correlation functions have the correct behaviour in the short-time and hydrodynamic limits have been given by Forster and Martin (1970) for a low density, weak coupling model and generalized independently by Mazenko (1974) and Forster (1974). The correlation functions, kinetic equation and these conditions are defined in § 2. In § 3, we divide the collision integral  $\Sigma$  into two parts, the first part containing

information, such as sum rules, which is known and the second part being the remainder,  $\sigma$ , which is either entirely unknown or extremely difficult to analyse. The first part of the collision integral is formally eliminated from the kinetic equation and the correlation functions are written in terms of those matrix elements of  $\Sigma$  which are well defined in the short-time and hydrodynamic limits and reduced correlation functions which are solutions of the kinetic equation with the remainder of the collision integral  $\sigma$ . The general structure of the resulting correlation function is examined and this facilitates the comparison in § 4 of many existing theories. In § 5, interpolation formulae for the correlation functions are developed which give the correct short-time and hydrodynamic limits by keeping those parts which are known and approximating those which are unknown. The properties of the reduced correlation functions are determined and a simple approximation, namely the equation of Bhatnagar *et al* (1954, to be referred to as BGK) with a single relaxation time, is suggested.

The approach adopted here is similar to the recent work of Jhon and Forster (1975), the main difference being that we first eliminate formally the important matrix elements of the collision integral and examine the structure of the correlation functions, whereas the other authors start with an expansion of the collision integral in momentum space and introduce approximations before calculating the correlation functions.

## 2. Kinetic equation: properties of the collision integral

For a system of  $N$  particles, the  $i$ th particle being at  $\mathbf{r}_i(t)$  with momentum  $\mathbf{p}_i(t)$  at time  $t$ , the single-particle density is given by

$$f(1, t) = f(\mathbf{r}_1, \mathbf{p}_1, t) = \sum_{i=1}^N \delta(\mathbf{r}_1 - \mathbf{r}_i(t)) \delta(\mathbf{p}_1 - \mathbf{p}_i(t)). \quad (2.1)$$

The single-particle phase-space distribution function is defined as

$$S(1, t_1; 2, t_2) = S(\mathbf{r}_1, \mathbf{p}_1, t_1; \mathbf{r}_2, \mathbf{p}_2, t_2) = \langle \{f(1, t_1) - \langle f(1, t_1) \rangle\} \{f(2, t_2) - \langle f(2, t_2) \rangle\} \rangle, \quad (2.2)$$

where the angular brackets denote an average over an equilibrium ensemble, and from the space-time invariance of the system

$$S(\mathbf{r}_1, \mathbf{p}_1, t_1; \mathbf{r}_2, \mathbf{p}_2, t_2) \equiv S(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2; \mathbf{p}_1, \mathbf{p}_2).$$

We will follow the notation of Forster and Martin (1970) and Forster (1974), who use dimensionless momentum variables

$$\mathbf{p} = m v_0 \boldsymbol{\xi} \quad (2.3)$$

with

$$v_0^2 = 1/\beta m; \quad \beta = 1/k_B T,$$

and write the transform of  $S$  as

$$S(\mathbf{k}, z; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = i \int_0^\infty dt e^{izt} \int d\mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} S(\mathbf{r}, t; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2), \quad (2.4)$$

where

$$S(\mathbf{r}, t; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = (m v_0)^6 S(\mathbf{r}, t; \mathbf{p}_1, \mathbf{p}_2).$$

The transformed distribution function is the solution of the kinetic equation (Forster 1974)

$$(z - v_0 \mathbf{k} \cdot \boldsymbol{\xi}_1) S(\mathbf{k}, z; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = -S_0(\mathbf{k}; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) + \int d\boldsymbol{\xi}_3 \Sigma(\mathbf{k}, z; \boldsymbol{\xi}_1, \boldsymbol{\xi}_3) S(\mathbf{k}, z; \boldsymbol{\xi}_3, \boldsymbol{\xi}_2) \quad (2.5)$$

where

$$S_0(\mathbf{k}; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = S(\mathbf{k}, t = 0; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = n\phi(\boldsymbol{\xi}_1)\delta(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) + n^2 h(k)\phi(\boldsymbol{\xi}_1)\phi(\boldsymbol{\xi}_2). \quad (2.6)$$

$\phi$  is the Maxwellian velocity distribution in dimensionless form

$$\phi(\boldsymbol{\xi}) = (2\pi)^{-3/2} \exp(-\frac{1}{2}\boldsymbol{\xi}^2), \quad (2.7)$$

$n$  is the density and  $h(k)$  is related to the liquid structure factor  $S(k)$  and direct correlation function  $c(k)$  by

$$S(k) = 1 + nh(k) = \frac{1}{1 - nc(k)}. \quad (2.8)$$

As is well known (Nelkin and Ranganathan 1967, Lebowitz *et al* 1969, Sykes 1973), the collision integral  $\Sigma$  separates into two parts

$$\Sigma(\mathbf{k}, z; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \Sigma^s(\mathbf{k}, \boldsymbol{\xi}_1) + \Sigma^c(\mathbf{k}, z; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \quad (2.9)$$

with the static part being

$$\Sigma^s(\mathbf{k}, \boldsymbol{\xi}_1) = -nc(k)v_0 \mathbf{k} \cdot \boldsymbol{\xi}_1 \phi(\boldsymbol{\xi}_1). \quad (2.10)$$

In order to discuss the properties of the collision integral  $\Sigma^c$ , Forster and Martin (1970) and Forster (1974) introduce a complete set of states  $X_\nu(\boldsymbol{\xi})$ ; the first five, which we will call the hydrodynamic states, are

$$\begin{aligned} X_1(\boldsymbol{\xi}) &= 1; & X_2(\boldsymbol{\xi}) &= \boldsymbol{\xi} \cdot \hat{\mathbf{k}}; & X_3(\boldsymbol{\xi}) &= (\boldsymbol{\xi}^2 - 3)/\sqrt{6} \\ X_4(\boldsymbol{\xi}) &= \boldsymbol{\xi} \cdot \hat{\mathbf{i}}; & X_5(\boldsymbol{\xi}) &= \boldsymbol{\xi} \cdot \hat{\mathbf{j}} \end{aligned} \quad (2.11)$$

with  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$  Cartesian unit vectors perpendicular to  $\hat{\mathbf{k}}$ . The remaining states will be left unspecified; they can for example be taken as Hermite polynomials. These states are orthogonal and normalized with weight function  $\phi$ :

$$\langle \nu | \mu \rangle = \int d\boldsymbol{\xi} X_\nu(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}) X_\mu(\boldsymbol{\xi}) = \delta_{\nu, \mu}. \quad (2.12)$$

We also define matrix elements by

$$\langle \nu | F(\mathbf{k}, z) | \mu \rangle = F_{\nu\mu}(\mathbf{k}, z) = \int d\boldsymbol{\xi}_1 \int d\boldsymbol{\xi}_2 X_\nu(\boldsymbol{\xi}_1) F(\mathbf{k}, z; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \phi(\boldsymbol{\xi}_2) X_\mu(\boldsymbol{\xi}_2) \quad (2.13)$$

where again  $\phi$  is the Maxwellian function given in equation (2.7).

The correlation functions we are particularly interested in are the density-density function

$$g_{nn}(\mathbf{k}, z) = \int d\boldsymbol{\xi}_1 \int d\boldsymbol{\xi}_2 S(\mathbf{k}, z; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \quad (2.14)$$

and the current-current function

$$\begin{aligned} \mathbf{g}(\mathbf{k}, z) &= (mv_0)^2 \int d\xi_1 \int d\xi_2 \xi_1 S(\mathbf{k}, z; \xi_1, \xi_2) \xi_2 \\ &\equiv g_l(k, z) \frac{\mathbf{k}\mathbf{k}}{k^2} + g_t(k, z) \left( \mathbf{1} - \frac{\mathbf{k}\mathbf{k}}{k^2} \right) \end{aligned} \tag{2.15}$$

where  $g_l$  and  $g_t$  are the longitudinal and transverse parts respectively. If we now put

$$S(\mathbf{k}, z; \xi_1, \xi_2) = \bar{S}(\mathbf{k}, z; \xi_1, \xi_2) \phi(\xi_2) \tag{2.16}$$

and

$$S_0(\mathbf{k}; \xi_1, \xi_2) = \bar{S}_0(\mathbf{k}; \xi_1, \xi_2) \phi(\xi_2), \tag{2.17}$$

so that the kinetic equation (2.5) is satisfied by  $\bar{S}$  and  $\bar{S}_0$ , then the correlation functions of interest can be written in the compact form

$$\begin{aligned} g_{nn}(k, z) &= \langle 1 | \bar{S}(k, z) | 1 \rangle \\ g_l(k, z) &= \langle 2 | \bar{S}(k, z) | 2 \rangle (mv_0)^2 \\ g_t(k, z) &= \langle 4 | \bar{S}(k, z) | 4 \rangle (mv_0)^2 \equiv \langle 5 | \bar{S}(k, z) | 5 \rangle (mv_0)^2. \end{aligned} \tag{2.18}$$

The last equality and the additional relationships

$$\begin{aligned} \langle 4 | \bar{S}(k, z) | 2 \rangle &\equiv \langle 5 | \bar{S}(k, z) | 2 \rangle \equiv \langle 5 | \bar{S}(k, z) | 4 \rangle \equiv \langle 2 | \bar{S}(k, z) | 4 \rangle \equiv \langle 2 | \bar{S}(k, z) | 5 \rangle \\ &\equiv \langle 4 | \bar{S}(k, z) | 5 \rangle \equiv 0 \end{aligned} \tag{2.19}$$

follow from the form of equation (2.15).

We now turn to the properties of the collision integral  $\Sigma^c$  as derived by Forster and Martin (1970) and Forster (1974). We start by expanding  $\Sigma^c$  in terms of the states  $X$  as

$$\Sigma^c(\mathbf{k}, z; \xi_1, \xi_2) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Sigma_{nm}^c(k, z) X_n(\xi_1) \phi(\xi_1) X_m(\xi_2) \tag{2.20}$$

where

$$\Sigma_{\nu\mu}^c(k, z) = \langle \nu | \Sigma^c(k, z) | \mu \rangle. \tag{2.21}$$

From the condition of detailed balance

$$\Sigma^c(\mathbf{k}, z; \xi_1, \xi_2) \phi(\xi_2) = \Sigma^c(\mathbf{k}, z; \xi_2, \xi_1) \phi(\xi_1)$$

it follows that

$$\Sigma_{\nu\mu}^c(k, z) = \Sigma_{\mu\nu}^c(k, z). \tag{2.22}$$

Furthermore, from the conservation of particle number,

$$\int d\xi_1 \Sigma^c(\mathbf{k}, z; \xi_1, \xi_2) = 0$$

so that

$$\Sigma_{1\mu}^c(k, z) = 0 \quad \text{for all } \mu. \tag{2.23}$$

2.1. Short-time limit

In Forster's (1974) notation, as  $z \rightarrow \infty$ :

$$\Sigma^c(k, z; \xi_1, \xi_2) = \frac{1}{z} \Sigma_0^c(k; \xi_1, \xi_2) + \frac{1}{z^2} \Sigma_1^c(k; \xi_1, \xi_2) + O\left(\frac{1}{z^3}\right). \quad (2.24)$$

The matrix elements of  $\Sigma_0^c$  defined by

$$\Sigma_{\nu\mu}^c(k)_0 = \langle \nu | \Sigma_0^c(k) | \mu \rangle$$

involve the sum rules

$$\omega_E^2 = \frac{n}{m} \int d\mathbf{r} g(r) \nabla^2 \Phi(r), \quad (2.25)$$

$$\omega_1^2 = 3v_0^2 k^2 + \frac{n}{m} \int d\mathbf{r} g(r) (1 - \cos \mathbf{k} \cdot \mathbf{r}) (\hat{\mathbf{k}} \cdot \nabla)^2 \Phi(r) \quad (2.26)$$

and

$$\omega_t^2 = v_0^2 k^2 + \frac{n}{2m} \int d\mathbf{r} g(r) (1 - \cos \mathbf{k} \cdot \mathbf{r}) [\nabla^2 - (\hat{\mathbf{k}} \cdot \nabla)^2] \Phi(r). \quad (2.27)$$

Here  $\Phi(r)$  is the two-body potential and  $g(r)$  the radial distribution function, both of which are spherically symmetric for a liquid. The results are

$$\begin{aligned} \Sigma_{22}^c(k)_0 &= \omega_1^2 - v_0^2 k^2 (3 - nc(k)) \\ \Sigma_{33}^c(k)_0 &= \frac{2}{3} \omega_E^2 \\ \Sigma_{44}^c(k)_0 &\equiv \Sigma_{55}^c(k)_0 = \omega_t^2 - v_0^2 k^2 \end{aligned} \quad (2.28)$$

and all the other matrix elements involving at least one hydrodynamic state, that is  $\nu$  and/or  $\mu$  equal to 2, 3, 4 and 5, vanish. The remaining elements are given by

$$\Sigma_{\nu\mu}^c(k)_0 = \frac{1}{3} \omega_E^2 \int d\xi_1 \phi(\xi_1) \frac{\partial X_\nu(\xi_1)}{\partial \xi_1} \cdot \frac{\partial X_\mu(\xi_1)}{\partial \xi_1}. \quad (2.29)$$

Forster (1974) also considers the second term  $\Sigma_1^c$  in equation (2.24) but this is not required for our purposes.

2.2. Hydrodynamic limit

In the hydrodynamic limit,  $z$  and  $k \rightarrow 0$ , the matrix elements for the hydrodynamic states can be expanded to third order in  $z$  and  $k$  as (Forster and Martin 1970, Mazenko 1974, Forster 1974):

$$\begin{aligned} \Sigma_{22}^c(k, z) &= -i\gamma_{22} v_0^2 k^2, \\ \Sigma_{23}^c(k, z) &= \frac{2}{\sqrt{6}} \left( \frac{3}{4} \alpha v_0 k + i\gamma_{23} z v_0 k \right), \\ \Sigma_{33}^c(k, z) &= -\bar{\alpha} z - i \left( \frac{2}{3} \gamma_{33} v_0^2 k^2 + \bar{\gamma}_{33} z^2 \right) \\ \Sigma_{44}^c(k, z) &\equiv \Sigma_{55}^c(k, z) = -i\gamma_t v_0^2 k^2. \end{aligned} \quad (2.30)$$

To this order, the longitudinal and transverse modes are decoupled so that

$$\Sigma_{24}^c = \Sigma_{25}^c = \Sigma_{34}^c = \Sigma_{35}^c = \Sigma_{45}^c = 0. \tag{2.31}$$

By comparing these expansions with the exact results in the hydrodynamic limit, information can be obtained about the equilibrium properties as well as the transport coefficients of the liquid. The equilibrium properties of the correlation functions are given correctly provided  $\alpha$  and  $\bar{\alpha}$  are chosen so that

$$\left(\frac{\partial P}{\partial T}\right)_n = nk_B(1 + \frac{3}{4}\alpha) \tag{2.32}$$

and

$$mc_v = \frac{3}{2}k_B(1 + \bar{\alpha}), \tag{2.33}$$

where  $k_B$  is Boltzmann's constant.

The viscosity, thermal conductivity and bulk viscosity are respectively

$$\begin{aligned} \eta &= mnv_0^2\gamma_t + \eta' \\ \kappa &= nk_Bv_0^2\gamma_{33} + \kappa' \end{aligned} \tag{2.34}$$

and

$$\zeta = \frac{1}{3}mnv_0^2(3\gamma_{22} - 4\delta\gamma_{23} + 2\delta^2\bar{\gamma}_{33} - 4\gamma_t) + \zeta'$$

where

$$\delta = (1 + 3\alpha/4)/(1 + \bar{\alpha}).$$

$\eta'$ ,  $\kappa'$  and  $\zeta'$  are given as complicated matrix elements involving non-hydrodynamic states by Forster (1974) and will not be written out here.

### 3. Separation of the collision integral

For a theory of correlation functions to be correct in both the short-time and hydrodynamic limits and to interpolate reasonably between them, it is clear from the previous section that the important matrix elements of the collision integral  $\Sigma^c$  are those between the hydrodynamic states. In this section we will eliminate these matrix elements from the collision integral and kinetic equation and investigate the general structure of the correlation functions.

To illustrate the method, which is a generalization of the work of Lebowitz *et al* (1969), we first in § 3.1 eliminate just the static term  $\Sigma^s$  from the kinetic equation and then remove  $\Sigma_{22}^c$  in § 3.2 and finally all the matrix elements between hydrodynamic states in § 3.3.

#### 3.1. Static term

With the explicit form for  $\Sigma^s$  given in equation (2.10), the kinetic equation for  $\bar{S}$  is

$$\begin{aligned} (z - v_0kX_2(\xi_1))\bar{S}(k, z; \xi_1, \xi_2) &= -\bar{S}_0(k; \xi_1, \xi_2) - nc(k)v_0kX_2(\xi_1)\phi(\xi_1) \int d\xi_3 \bar{S}(k, z; \xi_3, \xi_2) \\ &+ \int d\xi_3 \Sigma^c(k, z; \xi_1, \xi_3)\bar{S}(k, z; \xi_3, \xi_2). \end{aligned} \tag{3.1}$$

Suppose, as with Lebowitz *et al* (1969),  $G_0$  is the solution of

$$(z - v_0 k X_2(\xi_1))G_0(k, z; \xi_1, \xi_2) = -\delta(\xi_1 - \xi_2) + \int d\xi_3 \Sigma^c(k, z; \xi_1, \xi_3)G_0(k, z; \xi_3, \xi_2). \quad (3.2)$$

From equation (2.22), we find

$$G_0(k, z; \xi_1, \xi_2)\phi(\xi_2) = G_0(k, z; \xi_2, \xi_1)\phi(\xi_1)$$

so that

$$\langle \nu | G_0(k, z) | \mu \rangle = \langle \mu | G_0(k, z) | \nu \rangle \quad (3.3)$$

and from equation (2.23)

$$z \langle 1 | G_0(k, z) | \mu \rangle - v_0 k \langle 2 | G_0(k, z) | \mu \rangle = -\delta_{\mu,1}. \quad (3.4)$$

The distribution functions  $\bar{S}$  and  $G_0$  are related by

$$\begin{aligned} \bar{S}(k, z; \xi_1, \xi_2) &= \int d\xi_3 G_0(k, z; \xi_1, \xi_3) \\ &\quad \times \left[ \bar{S}_0(k; \xi_3, \xi_2) + nc(k)v_0 k X_2(\xi_3)\phi(\xi_3) \int d\xi_4 \bar{S}(k, z; \xi_4, \xi_2) \right] \\ &= nG_0(k, z; \xi_1, \xi_2) + n^2 h(k) \int d\xi_3 G_0(k, z; \xi_1, \xi_3)\phi(\xi_3) \\ &\quad + nc(k)v_0 k \int d\xi_3 G_0(k, z; \xi_1, \xi_3) X_2(\xi_3)\phi(\xi_3) \int d\xi_4 \bar{S}(k, z; \xi_4, \xi_2) \end{aligned} \quad (3.5)$$

so that the matrix elements of  $\bar{S}$  are given by

$$\langle \nu | \bar{S}(k, z) | \mu \rangle = n(1 + nh(k)\delta_{\mu,1})\langle \nu | G_0(k, z) | \mu \rangle + nc(k)v_0 k \langle \nu | G_0(k, z) | 2 \rangle \langle 1 | \bar{S}(k, z) | \mu \rangle.$$

By setting  $\nu = 1$ ,  $\langle 1 | \bar{S} | \mu \rangle$  can be calculated and then used to obtain all the other matrix elements  $\langle \nu | \bar{S} | \mu \rangle$ .

For the correlation functions defined in equation (2.18), the results are

$$\frac{g_{nn}(k, z)}{nS(k)} = \frac{\langle 1 | G_0(k, z) | 1 \rangle}{1 - nc(k) - znc(k)\langle 1 | G_0(k, z) | 1 \rangle} \quad (3.6)$$

$$\frac{g_1(k, z)}{nS(k)} = m^2 \frac{z}{k^2} \left( z \frac{g_{nn}(k, z)}{nS(k)} + 1 \right) \quad (3.7)$$

$$\frac{g_t(k, z)}{n} = (mv_0)^2 \langle 4 | G_0(k, z) | 4 \rangle. \quad (3.8)$$

Here, we have used

$$\langle 2 | G_0(k, z) | 4 \rangle = \langle 2 | G_0(k, z) | 5 \rangle = \langle 4 | G_0(k, z) | 5 \rangle \equiv 0 \quad (3.9)$$

and

$$\langle 4 | G_0(k, z) | 4 \rangle = \langle 5 | G_0(k, z) | 5 \rangle \quad (3.10)$$

in order to maintain the structure of equation (2.15).



The result for  $g_{nn}$  in equation (3.6) can be used to derive a mean-field theory expression for the response function in the form

$$\chi(k, z) = \frac{z}{n} g_{nn}(k, z) + S(k) = v_0 k \frac{1}{n} \langle 1 | \bar{S}(k, z) | 2 \rangle \equiv \frac{\chi_{sc}(k, z)}{1 - \psi(k, z) \chi_{sc}(k, z)}, \tag{3.11}$$

where the screened response function

$$\chi_{sc}(k, z) = v_0 k \langle 1 | G_0(k, z) | 2 \rangle \tag{3.12}$$

and the polarization potential

$$\psi(k, z) = nc(k). \tag{3.13}$$

Nelkin (1969) has shown that several of the older theories (Nelkin and Ranganathan 1967, Singwi *et al* 1970, Kerr 1968) can be cast in this form with particular choices for  $\psi$  and  $\chi_{sc}$  so we will not discuss it further. Finally, we note that the relationship between  $g_{nn}$  and  $g_l$  in equation (3.7) following from particle conservation is well known.

### 3.2. $\Sigma^s$ and $\Sigma_{22}^c$

We now eliminate from the kinetic equation the static and  $\Sigma_{22}^c$  terms. As the method is the same as in § 3.1, the details will be kept to a minimum.

Put

$$\Sigma^c(k, z; \xi_1, \xi_2) = \Sigma_{22}^c(k, z) X_2(\xi_1) \phi(\xi_1) X_2(\xi_2) + \sigma_2(k, z; \xi_1, \xi_2) \tag{3.14}$$

and suppose that  $G_2$  is the solution of the reduced kinetic equation

$$(z - v_0 k X_2(\xi_1)) G_2(k, z; \xi_1, \xi_2) = -\delta(\xi_1 - \xi_2) + \int d\xi_3 \sigma_2(k, z; \xi_1, \xi_3) G_2(k, z; \xi_3, \xi_2). \tag{3.15}$$

As before

$$\langle \nu | G_2(k, z) | \mu \rangle = \langle \mu | G_2(k, z) | \nu \rangle \tag{3.16}$$

and

$$z \langle 1 | G_2(k, z) | \mu \rangle - v_0 k \langle 2 | G_2(k, z) | \mu \rangle = -\delta_{\mu,1}.$$

The distribution functions  $\bar{S}$  and  $G_2$  are related by

$$\begin{aligned} \bar{S}(k, z; \xi_1, \xi_2) &= \int d\xi_3 G_2(k, z; \xi_1, \xi_3) \\ &\times \left( \bar{S}_0(k; \xi_3, \xi_2) + nc(k) v_0 k X_2(\xi_3) \phi(\xi_3) \int d\xi_4 \bar{S}(k, z; \xi_4, \xi_2) \right. \\ &\left. - \Sigma_{22}^c(k, z) X_2(\xi_3) \phi(\xi_3) \int d\xi_4 X_2(\xi_4) \bar{S}(k, z; \xi_4, \xi_2) \right) \end{aligned} \tag{3.17}$$

from which the matrix elements of  $\bar{S}$  and hence the correlation functions can be determined.

In this case, the mean-field theory of equation (3.11) is maintained but with

$$\chi_{sc}(k, z) = v_0 k \langle 1 | G_2(k, z) | 2 \rangle \quad (3.18)$$

and

$$\psi(k, z) = nc(k) - \frac{z}{(v_0 k)^2} \Sigma_{22}^c(k, z). \quad (3.19)$$

The longitudinal correlation  $g_1$  again satisfies equation (3.7) and the transverse correlation is now

$$\frac{g_i(k, z)}{n} = (mv_0)^2 \langle 4 | G_2(k, z) | 4 \rangle \quad (3.20)$$

with the same assumptions for  $G_2$  as equations (3.9) and (3.10) for  $G_0$ .

We defer until the next section the discussion of these and the following correlation functions.

### 3.3. $\Sigma^s$ and hydrodynamic matrix elements

We put

$$\Sigma^c(k, z; \xi_1, \xi_2) = \sum_{n=2}^5 \sum_{m=2}^5 \Sigma_{nm}^c(k, z) X_n(\xi_1) \phi(\xi_1) X_m(\xi_2) + \sigma(k, z; \xi_1, \xi_2) \quad (3.21)$$

and let  $G(k, z; \xi_1, \xi_2)$  be the solution of the kinetic equation with collision integral  $\sigma(k, z; \xi_1, \xi_2)$ . Then the matrix elements

$$\bar{S}_{\mu\nu} = \bar{S}_{\mu\nu}(k, z) = \langle \mu | \bar{S}(k, z) | \nu \rangle$$

for the hydrodynamic states are solutions of the equations

$$\sum_{m=1}^5 (\delta_{\mu m} + A_{\mu m}) \bar{S}_{m\nu} = a_{\mu\nu}$$

where

$$A_{\mu 1} = -nc(k) v_0 k G_{\mu 2}$$

$$A_{\mu m} = \sum_{n=2}^5 G_{\mu n} \Sigma_{nm}^c \quad \text{for } 2 \leq m \leq 5$$

and

$$a_{\mu\nu} = n(1 + nh(k)\delta_{\nu,1}) G_{\mu\nu}.$$

If there is no coupling between the longitudinal (1, 2, 3) and transverse (4, 5) states, then

$$\chi(k, z) = \frac{\chi_0(k, z)}{\chi_1(k, z) - \psi(k, z)\chi_0(k, z)} \quad (3.22)$$

where

$$\begin{aligned} \chi_0(k, z) &= v_0 k [G_{12} + \Sigma_{33}^c (G_{12} G_{33} - G_{13} G_{32})] \\ \chi_1(k, z) &= (1 + \Sigma_{23}^c G_{23})^2 + G_{33} (\Sigma_{33}^c - \Sigma_{23}^c \Sigma_{23}^c G_{22}) \end{aligned} \quad (3.23)$$

and  $\psi$  is given in equation (3.19); the relationship (3.7) is maintained and

$$\frac{1}{n} g_t(k, z) = (mv_0)^2 \frac{G_{44}}{1 + \Sigma_{44}^c G_{44}}. \quad (3.24)$$

Obviously this method can be extended and matrix elements for non-hydrodynamic states formally eliminated once these states have been specified.

#### 4. Comparison of theories

In this section we will compare many of the existing theories either by calculating the matrix elements of the collision integral when approximate kinetic equations have been used or by casting the results into one or other of the general forms of § 3.1, § 3.2 and § 3.3. The theories will be discussed briefly and in chronological order, the reader being referred to the original papers or the review article (Schofield 1975) for further details.

##### 4.1. Hubbard and Beeby

Unlike most other theories, Hubbard and Beeby (1969) try to calculate the equilibrium properties of the liquid as well as the correlation functions. Their results for  $\chi(k, z)$ , equation (3.11), can be written in the form of § 3.2 with

$$\chi_{sc}(k, z) = (v_0 k)^2 Q(k, z) \quad (4.1)$$

and

$$\psi(k, z) = nc(k) - \frac{z}{(v_0 k)^2} \Sigma_{22}^c(k, z) \quad (4.2)$$

where  $\Sigma_{22}^c$  is approximated by

$$\Sigma_{22}^c(k, z) = \frac{1}{z} \Sigma_{22}^c(k)_0.$$

$\Sigma_{22}^c(k)_0$  is given in equation (2.28) and  $Q(k, z)$  is related to the self or incoherent correlation function and will not be written out.

##### 4.2. Chung and Yip

For the longitudinal correlation function, Chung and Yip (1969) use an approximation with two relaxation times; a similar theory with a single relaxation time will be discussed below in § 4.6. For the transverse correlation function, a single relaxation time description is used, the result being equivalent to § 3.3, equation (3.24), with

$$\frac{1}{G_{44}(k, z)} = -z - \frac{i\tau}{1 - iz\tau} (v_0 k)^2 \quad (4.3)$$

and

$$\Sigma_{44}^c(k, z) = \frac{-i\tau}{1 - iz\tau} \Sigma_{44}^c(k)_0. \quad (4.4)$$

We will show in the appendix how  $G_{44}$  can be derived from the BGK equation.

4.3. *Lebowitz, Percus and Sykes*

The collision integral in the short-time kinetic equation of Lebowitz *et al* (1969) has matrix elements

$$\Sigma_{\mu\nu}^c(k, z) = -i\tilde{w}(-iz)\Sigma_{\mu\nu}^c(k)_0 \tag{4.5}$$

where the function  $\tilde{w}$  is unspecified apart from satisfying the condition

$$\tilde{w}(s) \rightarrow 1/s \quad \text{as } s \rightarrow \infty.$$

When the hydrodynamic matrix elements are eliminated, the reduced kinetic equation is the Fokker-Planck equation with a modified friction coefficient.

4.4. *Akcasu and Duderstadt*

The approximation developed independently by Akcasu and Duderstadt (1969, 1970) is the same as § 4.3 but with the explicit choice

$$\tilde{w}(s) = \frac{1}{s + \alpha(k)}.$$

If we introduce a relaxation time  $\tau$  by putting  $\alpha = 1/\tau$ , then equation (4.5) becomes

$$\Sigma_{\mu\nu}^c(k, z) = \frac{-i\tau}{1 - iz\tau} \Sigma_{\mu\nu}^c(k)_0. \tag{4.6}$$

4.5. *Pathak and Singwi*

The screened field theory of Pathak and Singwi (1970) gives a response function

$$\chi_{ps}(k, z) = \frac{\chi_{sc}^{ps}(k, z)}{1 - \psi_{ps}(k)\chi_{sc}^{ps}(k, z)} \tag{4.7}$$

where  $\chi_{sc}^{ps}(k, z)$  is the response function for a modified ideal gas and involves a mean free path  $\Gamma(k)$ . The comparison of their result with (3.11) is not entirely trivial because  $\chi_{sc}$  as defined in (3.12) is obtained from a reduced kinetic equation whereas  $\chi_{sc}^{ps}$  in (4.7) can be regarded as the response function calculated from a non-reduced but approximate kinetic equation. To effect the comparison, we put

$$\chi = \chi_{sc}/(1 - \psi\chi_{sc})$$

as before and

$$\chi_0 = \chi_{sc}/(1 - \psi_0\chi_{sc})$$

so that

$$\chi = \frac{\chi_0}{1 - (\psi - \psi_0)\chi_0}. \tag{4.8}$$

Then, allowing for minor differences in notations,

$$\begin{aligned} \chi(k, z) &= -\frac{1}{\beta n} \chi_{ps}(k, z) \\ \chi_0(k, z) &= -\frac{1}{\beta n} \chi_{sc}^{ps}(k, z) \end{aligned} \tag{4.9}$$

and

$$\psi(k, z) - \psi_0(k, z) = -\beta n \psi_{ps}(k).$$

$\psi$  and  $\psi_0$  can now be chosen so that

$$\psi(k, z) = nc(k) - \frac{z}{(v_0 k)^2} \Sigma_{22}^c(k, z) \quad (4.10)$$

with the approximation

$$\Sigma_{22}^c(k, z) = \frac{1}{z} \Sigma_{22}^c(k)_0$$

and

$$\psi_0(k, z) = -\frac{3}{2} \frac{\Gamma(k)}{(v_0 k)^2} = -\frac{1}{(v_0 k)^2} \left( \frac{m}{nk^2} \langle \omega_{ps} \rangle_3 - 3(v_0 k)^2 \right). \quad (4.11)$$

The last expression in the notation of Pathak and Singwi is analogous to (4.10) for their modified ideal gas and obviously vanishes when  $\Gamma$  is zero.

#### 4.6. Lovesey

The theory of Lovesey (1971) for the longitudinal correlation function is similar to that of Chung and Yip but involves only a single relaxation time. The result for  $\chi$  is of the form of (3.11) with

$$\frac{1}{\chi_{sc}} = \frac{1}{v_0 k G_{12}} = -\left( \frac{z}{v_0 k} \right)^2 + \frac{1 - 3iz\tau}{1 - iz\tau} \quad (4.12)$$

and  $\psi$  given by (3.19) with

$$\Sigma_{22}^c(k, z) = \frac{-i\tau}{1 - iz\tau} \Sigma_{22}^c(k)_0. \quad (4.13)$$

We will show in the appendix that (4.12) can be deduced from a simplified form of the BCK equation.

#### 4.7. Kugler

Kugler (1973) considers

$$\chi_k(k, z) = \frac{\chi_{k0}(k, z)}{1 + \phi_k(k, z) \chi_{k0}(k, z)} \quad (4.14)$$

and takes  $\chi_{k0}$  to be the ideal gas response function. As can be seen from § 4.5 when  $\Gamma = 0$ , no additional manipulation is required and (4.14) is equivalent to (3.11) with

$$\begin{aligned} \chi(k, z) &= \frac{1}{\beta n} \chi_k(k, z), \\ \chi_{sc}(k, z) &= \frac{1}{\beta n} \chi_{k0}(k, z) \end{aligned} \quad (4.15)$$

and

$$\psi(k, z) = -\beta n \phi_k(k, z).$$

Kugler examines two choices for  $\phi_k$ , the simpler one leading to (3.19) with

$$\Sigma_{22}^c(k, z) = \frac{\mp i\tau}{1 \mp iz\tau} \Sigma_{22}^c(k)_0. \quad (4.16)$$

#### 4.8. Jhon and Forster

Jhon and Forster (1975) consider the formal expansion of the collision integral in momentum space and approximate the first few terms. In our notation, with the coordinate axis  $z$  chosen to be parallel to  $k$ , the matrix elements of their collision integral are

$$\Sigma_{\mu\nu}^c(k, z) = \sigma \Sigma_{\mu\nu}^c(k)_0 + A [(\hat{\sigma} - \sigma)(v_0 k)^2 \delta_{\mu,\nu} \delta_{\nu,2} - \hat{\sigma} v_0 k (B(\nu) \delta_{\mu,2} + B(\mu) \delta_{\nu,2}) + \hat{\sigma} B(\mu) B(\nu)]. \quad (4.17)$$

Here

$$\begin{aligned} \sigma &= 1/(z + i/\tau) \\ \hat{\sigma} &= 1/(z - \omega_\epsilon^2 \sigma) \\ A &= \frac{\beta \tilde{\chi}_{\tau\epsilon}^2(k)}{n \tilde{\chi}_{\epsilon\epsilon}(k)} \end{aligned} \quad (4.18)$$

and

$$B(\mu) = A_i \langle \mu | \xi_i^2 - 1 \rangle$$

where

$$A_i = \frac{1}{2} \sigma \left( \frac{\beta}{n} \tilde{\chi}_{\tau\epsilon}(k) \right)^{-1} \hat{v}_i(k).$$

and  $\hat{v}_i$ , which is related to the short-time behaviour, will not be written out. In particular,

$$\Sigma_{22}^c(k, z) = \sigma \Sigma_{22}^c(k)_0 + A (\hat{\sigma} - \sigma)(v_0 k)^2 \quad (4.19)$$

and

$$\Sigma_{44}^c(k, z) = \sigma \Sigma_{44}^c(k)_0 \quad (4.20)$$

The first term in (4.17) is similar to the theories in § 4.3 and § 4.4 and by choosing Hermite polynomials, Jhon and Forster solve the kinetic equation and calculate the correlation functions. There remains the choice of the functions  $\omega_\epsilon(k)$ ,  $\tilde{\chi}_{\epsilon\epsilon}(k)$ ,  $\tilde{\chi}_{\tau\epsilon}(k)$  and  $\tau(k)$  and these are approximated.

#### 4.9. Summary

This analysis puts into perspective the various theories so far proposed for the correlation functions. For the hydrodynamic matrix elements such as  $\Sigma_{22}$  the simplest choice is the short-time limit

$$\Sigma_{22}^c(k, z) = \frac{1}{z} \Sigma_{22}^c(k)_0.$$

To include a better description of the  $z$  dependence nearly all authors have used a relaxation time  $\tau$  with

$$\Sigma_{22}^c(k, z) = \frac{-i\tau}{1 - iz\tau} \Sigma_{22}^c(k)_0, \quad (4.21)$$

although the methods suggested for determining  $\tau$  which may be  $k$  dependent have varied widely. For the reduced correlation functions and  $\chi_{sc}$  much greater imagination has been shown and approximations have been developed based on the ideal gas, the self function, and the Fokker–Planck and BGK equations. Clearly, the recent theory of Jhon and Forster is more ambitious than the others.

## 5. Interpolation formulae

In order to develop interpolation formulae for the correlation functions which are correct in both the hydrodynamic and short-time limits, we must consider the matrix elements of the collision integral between hydrodynamic states and the reduced correlation functions which occur in equations (3.22) and (3.24). Here we will adopt a pragmatic point of view and approximate these functions separately.

The behaviour of the hydrodynamic matrix elements has been discussed in § 2. For example,

$$\Sigma_{22}^c(k, z) \rightarrow \frac{1}{z} \Sigma_{22}^c(k)_0 \quad \text{as } z \rightarrow \infty \quad (5.1)$$

with  $\Sigma_{22}^c(k)_0$  defined in terms of equilibrium functions in equation (2.28) and

$$\Sigma_{22}^c(k, z) \rightarrow -i\gamma_{22}(v_0k)^2 \quad \text{as } k, z \rightarrow 0 \quad (5.2)$$

where  $\gamma_{22}$  is constant and contributes to the bulk viscosity  $\zeta$  in equation (2.34). If we follow most other authors and put

$$\Sigma_{22}^c(k, z) = \frac{-i\tau}{1 - iz\tau} \Sigma_{22}^c(k)_0 \quad (5.3)$$

where  $\tau$  is a  $k$  dependent relaxation time, then condition (5.1) is automatically satisfied. Since

$$\Sigma_{22}^c(k)_0 = (v_0k)^2 \Sigma_0(k)$$

with  $\Sigma_0(k)$  remaining finite as  $k \rightarrow 0$ , then condition (5.2) follows from  $\tau(k) \rightarrow \tau(0)$  as  $k \rightarrow 0$  with

$$\gamma_{22} = \Sigma_0(0)\tau(0). \quad (5.4)$$

Thus both conditions can easily be satisfied and as the behaviour of  $\tau$  for non-zero  $k$  is arbitrary it can be chosen according to some other criterion such as the ideal gas limit. The other hydrodynamic matrix elements can be represented in a similar way.

The reduced correlation functions  $G_{\mu\nu}$  are solutions of a kinetic equation with the reduced collision integral  $\sigma$  defined in (3.21). If we take  $\sigma$  to be the Fokker–Planck operator as in the theories of § 4.3 and § 4.4, then the matrix elements of  $\sigma$  will be exact in the short-time limit, as in equation (2.29). However, the short-time behaviour of these matrix elements does not affect the short-time behaviour of the correlation functions of interest and so their exact representation is not necessary for our purposes. The obvious alternative to the Fokker–Planck equation is the BGK collision integral which has matrix elements

$$\sigma_{\mu\nu} = -\frac{i}{T} \delta_{\mu,\nu} (1 - \delta_{\nu,1} - \delta_{\nu,2} - \delta_{\nu,3} - \delta_{\nu,4} - \delta_{\nu,5}). \quad (5.5)$$

In general, the relaxation time  $T$  may be  $k$  dependent and different from  $\tau$  in (5.3). The defects of the BGK equation have been known for a long time: for example, the equilibrium properties of the correlation functions calculated from the BGK equation are the ideal gas results. (This can be seen directly from equations (2.32) and (2.33) by noting that  $\sigma_{23}(k, z)$  and  $\sigma_{33}(k, z)$  are zero by definition and hence  $\alpha$  and  $\bar{\alpha}$  are also zero from equation (2.30)). However, because of the elimination of the hydrodynamic matrix elements, the properties of the BGK equation are precisely those required here of the reduced kinetic equation.

If we choose simple representations such as (5.3) for the hydrodynamic matrix elements and calculate the reduced correlation functions from the BGK equation, the results for which are given in the appendix, then the correlation functions are completely determined, once the relaxation times are specified, and have the desired properties. Without writing out all the analysis, we find from equation (3.22) for the density–density correlation function

$$\frac{z}{n} g_{nn}(k, z) = -S(k) - \frac{1}{z^2}(v_0 k)^2 - \frac{1}{z^4}(v_0 k)^2 \omega_1^2 + O\left(\frac{1}{z^5}\right) \tag{5.6}$$

as  $z \rightarrow \infty$ , which is exact to fifth order. In the hydrodynamic limit,  $g_{nn}$  has the expected form (Kadanoff and Martin 1963) with

$$\begin{aligned} \frac{c_p}{c_v} &= 1 + \frac{2}{3} \frac{1}{(1 - nc(0))} \frac{(1 + 3\alpha/4)^2}{(1 + \bar{\alpha})} \\ c^2 &= \frac{c_p}{c_v} (1 - nc(0)) v_0^2 \\ \kappa &= nk_B v_0^2 (\gamma_{33} + \frac{5}{2} T_0) \\ \eta &= mn v_0^2 (\gamma_t + T_0) \\ \zeta &= \frac{1}{3} mn v_0^2 (3\gamma_{22} - 4\delta\gamma_{23} + 2\delta^2 \bar{\gamma}_{33} - 4\gamma_t). \end{aligned} \tag{5.7}$$

With the equations (2.32) and (2.33) and the result

$$1 - nc(0) = 1/S(0) = 1/nk_B T \kappa_T, \tag{5.8}$$

where  $\kappa_T$  is the isothermal compressibility, it can be seen that the first equation in (5.7) is an exact expression for the specific heat ratio  $c_p/c_v$  and the second equation is exact for the adiabatic sound velocity  $c$ . The transport coefficients are equivalent to (2.34) with the contributions  $\eta'$ ,  $\kappa'$  and  $\zeta'$  of the non-hydrodynamic states given by the corresponding BGK results. (The fact that these are not exact is not a serious defect of the theory because the experimental values of the transport coefficients cannot be separated unambiguously into parts and exact values assigned to  $\eta'$  etc.) The properties of the longitudinal function  $g_l$  follow immediately from equation (3.7) and the above results. Finally, for the transverse function, from equation (3.24)

$$\frac{z\beta}{nm} g_t(k, z) = -1 - \frac{1}{z^2} \omega_1^2 + O\left(\frac{1}{z^3}\right) \tag{5.9}$$

as  $z \rightarrow \infty$ .  $g_t$  can also be written as

$$g_t(k, z) = -\frac{nm}{\beta} \frac{1}{z + ik^2 \eta(k, z)/nm} \tag{5.10}$$



where  $\eta(k, z)$  is a generalized viscosity. Then, in the hydrodynamic limit

$$\eta(k, z) \rightarrow mnv_0^2(\gamma_t + T_0) \tag{5.11}$$

which is the same as the result in (5.7).

The theory outlined above is the simplest way of deriving interpolation formulae with the desired properties, although of course it is not unique and in fact the result for the transverse correlation function is almost identical to that of Chung and Yip in § 4.2. The main feature of this analysis is that, provided the correct structure is used for the correlation functions, the exact results in both the short-time and hydrodynamic limits can be obtained with only minimal restrictions on the relaxation times, namely that they have the correct values in the limit  $k \rightarrow 0$ . In order to make a sensible comparison with experiment, instead of merely curve fitting, it is clear that more information about the microscopic nature of the system has to be invoked to determine the  $k$  dependence of the relaxation times. Finally, the methods developed here are sufficiently flexible that they can easily be extended and collision integrals more complicated than the BGK equation used, as in the theory of Jhon and Forster.

**Appendix**

The collision integral of Bhatnagar *et al* (1954) is constructed so that it conserves particle number, momentum and kinetic energy. For a distribution function  $G$  their kinetic equation is

$$(z - v_0kX_2(\xi_1))G(k, z; \xi_1, \xi_2) = -\delta(\xi_1 - \xi_2) + \int d\xi_3 \sigma(k, z; \xi_1, \xi_3)G(k, z; \xi_3, \xi_2) \tag{A.1}$$

where the matrix elements of  $\sigma$  are

$$\sigma_{\mu\nu}(k, z) = -\epsilon \delta_{\mu,\nu} (1 - \delta_{\nu,1} - \delta_{\nu,2} - \delta_{\nu,3} - \delta_{\nu,4} - \delta_{\nu,5}) \tag{A.2}$$

and  $\epsilon$  is related to a relaxation time  $\tau_0$  by

$$\epsilon = i/\tau_0. \tag{A.3}$$

This kinetic equation can easily be solved and the correlation functions evaluated. In general

$$G_{\mu\nu} = G_{\mu\nu}(k, z) = \langle \mu | G(k, z) | \nu \rangle = G_{\nu\mu} \tag{A.4}$$

and

$$zG_{1\nu} - v_0kG_{2\nu} = -\delta_{1,\nu}. \tag{A.5}$$

The correlation functions for the hydrodynamic states are all of the form

$$G_{\mu\nu} = d_{\mu\nu}/\delta$$

and can be expressed in terms of a single integral

$$J = \frac{p}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{p-t} \tag{A.6}$$

where

$$p = \frac{1}{\sqrt{2}} \frac{z + \epsilon}{v_0 k}. \tag{A.7}$$

The results are

$$\begin{aligned} d_{12} &= \frac{1}{v_0 k} \left( 1 - J - \frac{1}{6} \frac{\epsilon}{z + \epsilon} [J + 4J(1 - J) + 2p^2(1 - J)] \right) \\ d_{13} &= \frac{1}{\sqrt{6}} \frac{1}{z + \epsilon} [J + 2p^2(1 - J)] \\ d_{33} &= \frac{1}{z + \epsilon} \left[ -\frac{2}{3} J \left( 1 - \frac{\epsilon}{z + \epsilon} J + \frac{z\epsilon}{(v_0 k)^2} (1 - J) \right) + \frac{1}{6} \left( \frac{z^2}{v_0^2 k^2} - 1 \right) [J + 2p^2(1 - J)] \right] \end{aligned} \tag{A.8}$$

$$\begin{aligned} d_{44} = d_{55} &= -\frac{J}{z + \epsilon(1 - J)} \delta \\ d_{14} = d_{15} = d_{34} = d_{35} = d_{45} &= 0 \end{aligned}$$

and

$$\delta = 1 - \frac{\epsilon}{z + \epsilon} J + \frac{z\epsilon}{(v_0 k)^2} (1 - J) + \epsilon d_{33}.$$

The other correlation functions are obtained from these by use of equations (A.4) and (A.5).

In the short-time limit,  $z \rightarrow \infty$  so that

$$p = \frac{z}{v_0 k \sqrt{2}} \left( 1 + \frac{\epsilon}{z} \right) \rightarrow \infty. \tag{A.9}$$

Likewise, in the hydrodynamic limit  $z \rightarrow 0$ ,  $k \rightarrow 0$  and

$$p = \frac{\epsilon}{v_0 k \sqrt{2}} \left( 1 + \frac{z}{\epsilon} \right) \rightarrow \infty$$

Now for large  $p$ :

$$J = 1 + \frac{1}{2p^2} + \frac{3}{4p^4} + O\left(\frac{1}{p^6}\right) \tag{A.10}$$

which enables the short-time and hydrodynamic limits of the correlation functions to be calculated. In particular, for large  $p$ ,

$$\frac{1}{G_{44}} = -\frac{[z + \epsilon(1 - J)]}{J} = -z + \frac{(v_0 k)^2}{z + \epsilon} [1 + O(1/p^2)]. \tag{A.11}$$

This is the approximation of Chung and Yip mentioned in § 4.2.

Finally, if a simplified form of the BGK equation is used with a collision integral which only conserves particle number and momentum in the  $\mathbf{k}$  direction, then

$$\sigma_{\mu\nu}(k, z) = -\epsilon \delta_{\mu,\nu} (1 - \delta_{\nu,1} - \delta_{\nu,2}).$$

In particular, we now find

$$\frac{1}{v_0 k G_{12}} = \frac{z\epsilon}{(v_0 k)^2} + 1 + \frac{z}{z+\epsilon} \frac{J}{1-J} = -\frac{z^2}{(v_0 k)^2} + 1 + \frac{2z}{z+\epsilon} [1 + O(1/p^2)]$$

which is the approximation of Lovesey in § 4.6.

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